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The Hierarchical Random Field Ising Model

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We introduce and solve explicitly a hierarchical approximation to the random field Ising model. This approximation is defined in terms of Peierls' contours. It exhibits a spontaneous magnetization in $d>2$ and illustrates some of the ideas used in the proof of that result for the real RFIM. In $d=2$, there is no spontaneous magnetization, but a very slow decay of correlations. However, we argue that this latter property is an artifact of the approximation. For the real RFIM, we expect exponential decay of correlations for any value of the disorder.

KEY WORDS: Random fields; renormalization group; Peierls contours.

1. INTRODUCTION

In this paper, we discuss some unresolved issues concerning the lower critical dimension d_l of the random field Ising model. The latter is defined by the Hamiltonian

$$\mathscr{H} = \frac{1}{2} \sum_{\langle xy \rangle} \sigma_x \sigma_y + \sum_x h_x \sigma_x \tag{1.1}$$

where $x \in \mathbf{Z}^d$, $\langle xy \rangle$ are nearest neighbor pairs, $\sigma_x = \pm 1$, and $\{h_x\}$ are independent random variables of mean zero ($\bar{h}_x = 0$) and variance ε^2 ($\bar{h}^2 = \varepsilon^2$).

It is known that, if $d \geqslant 3$, a phase transition occurs when ε is varied for $\beta = T^{-1}$ large. For ε small, ferromagnetism persists,[1,2] while for ε large,

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the spins tend to be aligned along the direction of the field.[3,4] This shows that, by definition, $d_l \leqslant 2$. The Imry–Ma argument,[5] the predictions of which are vindicated by those results in $d \geqslant 3$, suggests quite convincingly that in $d=2$, ferromagnetism will be destroyed for *any* $\varepsilon \neq 0$ (thus that $d_l = 2$). However, for reasons discussed below, this part of the Imry–Ma argument seems harder to prove than the results concerning the stability of ferromagnetic order in $d \geqslant 3$. Ironically, it is the $d \geqslant 3$ part of the argument that was challenged some time ago,[6] while the $d=2$ part seems to be generally accepted.

In this paper, we reconsider the Imry–Ma argument for $d=2$ and discuss the following question:

Does the correlation function $\overline{\langle \sigma_0 \sigma_y \rangle}$ (the bar denotes average over randomness, and $\langle \cdot \rangle$ is the thermal average) decay exponentially in x? It has been argued[7] that, at least in the limit $T=0$ and for ε small, the latter may decay only according to a power law, thus exhibiting a massless phase. Other authors[8] argue that there is exponential decay with a correlation length $\xi(\varepsilon) = \exp O(\varepsilon^{-2})$.

Although we cannot resolve rigorously this issue, we construct and solve a *hierarchical* model which approximates the RFIM and has the following properties:

1. There is long-range order in $d > 2$.
2. For $d=2$, there is no long-range order, but

$$\overline{\langle \sigma_0 \sigma_x \rangle} \simeq (\log \log |x|)^{-p} \qquad \text{for some} \quad p > 0 \tag{1.2}$$

so that the decay is very slow indeed. As we show, this model is a slight improvement over the Imry–Ma argument because it includes the influence of "contours within contours." The latter actually tend to lower the fluctuations of the field, because of a "screening" effect. However, we show that they do not lower those fluctuations enough to invalidate the main conclusion of the Imry–Ma argument, namely the absence of long-range order.

The hierarchical model can be described as follows: on each scale, the lattice is a disjoint union of blocks whose boundaries are the contours. L^d blocks on one scale form one block of the next scale. Contours are independent of each other in a given scale and also between scales. On the first scale each block is a site on which a random field lives. We obtain an exact recursion relation for the effective random fields on later scales. The variance of the field on the nth scale satisfies the recursion relation

$$\varepsilon_{n+1}^2 = \varepsilon_n^2 [1 - O(\varepsilon_n e^{-1/2\varepsilon_n^2})] \tag{1.3}$$

where the second term reflects the influence of contours (on the nth scale) within contours [on the $(n+1)$th scale]. Thus, the variance, constant to leading order (which is really the content of the Imry Ma argument) goes very slowly to zero $[\varepsilon_n^2 \sim (\ln n)^{-1}]$. This decrease of ε_n^2 is not rapid enough to produce long-range order (for comparison, when $d>2$, $\varepsilon_n^2 \sim L^{(2-d)n}$), but does cause the slow decay in (1.2).

Now one should ask: Do the recursion (1.3) and the decay (1.2) hold for the real (Ising) model? We believe that they do not and that they are basically an artifact of the hierarchical nature of the model. Indeed, by imposing that every contour be the boundary of a cube, we neglect the crucial fluctuations in the shapes of the contours. The real contours have lots of wiggles and may therefore take advantage of small fluctuations in the field. When these are taken into account, we find that the real recursion for ε_n appears to be (for ε small at least)

$$\varepsilon_{n+1}^2 = \varepsilon_n^2 + O(\varepsilon_n^4) \tag{1.4}$$

where the second term here is more relevant than the second term in (1.3) (see Section 3). Our arguments for (1.4) are very close to those used by Binder[9] when he studied the interface of the $d=2$ RFIM. See also ref. 10. From (1.4) one expects that ε grows under renormalization and since there certainly is exponential decay of correlations for ε large,[3,4] one expects a finite correlation length for all $\varepsilon \neq 0$ with $\xi(\varepsilon) \simeq \exp O(\varepsilon^{-2})$. However, this is generally hard to prove. Even if (1.4) can be rigorously established for ε small, it is difficult to control the flow of ε in the "intermediate" region, where ε is neither big nor small. This is a common problem encountered by rigorous renormalization group arguments. For instance, in the two-dimensional $O(N)$ rotator models for $N \geqslant 3$ one can compute the flow of the temperature under renormalization.[11] For small T, the system is driven toward higher temperatures, suggesting convergence toward the trivial fixed point $T=\infty$ and exponential decay of correlations for all $T \neq 0$. However, again, the intermediate region appears to be difficult to control. Pursuing this analogy, the Imry–Ma argument is similar to the Mermin–Wagner argument (except that the latter is rigorous): it indicates that there is no long-range order, but does not distinguish between exponential and nonexponential decay of correlations.

In the next section, we give some background on the renormalization group approach to the random field model and we formulate and solve the hierarchical model. In Section 3, we discuss the real model and argue that (1.4) should be the true flow of the variance of the field.

2. THE HIERARCHICAL RANDOM FIELD MODEL

We start by reviewing the contour representation of the RFIM and the renormalization group strategy of ref. 2. In so doing, we also recall the Imry–Ma argument. After that, we define and solve the hierarchical model.

Consider a box $V \subseteq \mathbf{Z}^d$ centered around the origin with $|V| = L^{dN}$. Throughout this paper, L is an arbitrary, but fixed, constant. Let us put + boundary conditions outside V. Then

$$\begin{aligned} Z^+(V) &= \sum_\sigma \exp[-\beta \mathscr{H}^+(\sigma)] \\ &= \text{const} \times \sum_\Gamma \prod_{\gamma \in \Gamma} e^{-\beta |\gamma|} e^{\beta[(h, V^+(\Gamma)) - (h, V^-(\Gamma))]} \end{aligned} \tag{2.1}$$

where the sum over Γ runs over *compatible* families of contours (i.e., different contours are disjoint and the signs that they determine match). $V^\pm(\Gamma)$ are the $\pm$ regions of V determined by Γ and $(h, \Lambda) \equiv \sum_{x \in \Lambda} h_x$. The constant in (2.1) is unimportant.

The two main steps in the renormalization group transformation of ref. 1 are:

1. Split the sum (2.1) into small and large contours. Small contours have a diameter less than L (and, moreover, all the fields within the contour are small; however, we shall not need this constraint in the explicitly solvable case below). Then, integrate out "explicitly" the small contours. This produces a new field, approximately equal to

$$H'_x = L^{1-d}\left(\sum_{y \in Lx} H_y + \text{free energy of small contours}\right) \tag{2.2}$$

where $x \in \mathbf{Z}^d$ indexes disjoint L^d blocks of the original lattice; Lx denotes the block indexed by x. The L^{1-d} factor in (2.2) is explained by the next step.

2. Block, or resum, the remaining large contours. Define the contours on the new lattice (composed of L^d blocks of the previous lattice) by

$$\gamma \to \gamma' = \{x \mid \gamma \cap Lx \neq \varnothing\} \tag{2.3}$$

and let the activities of the new contours γ' be (approximately) the sum of the ones of the old contours that are mapped onto γ':

$$\rho'(\gamma') = \sum_\Gamma \prod_{\gamma \in \Gamma} \rho(\gamma) \tag{2.4}$$

where $\rho(\gamma) = e^{-\beta|\gamma|}$ and the sum is over compatible Γ's s.t. $\gamma' = \{x \mid \Gamma \cap Lx \neq \varnothing\}$.

From the fact that contours are $(d-1)$-dimensional (hyper)surfaces one gets that

$$\rho'(\gamma') \leqslant e^{-\beta' |\gamma'|} \tag{2.5}$$

with $|\gamma'|$ = number of points in γ' (i.e., blocks in the original lattice) and

$$\beta' \cong L^{d-1}\beta \tag{2.6}$$

(This holds only for contours γ' that are the boundary of their interior. But here we shall not consider the other contours, ref. 2.) The L^{d-1} factor in (2.6) explains the L^{1-d} factor in (2.2), because we keep the β (or β') factor multiplying the field $[\beta' H'_x = \sum_{y \in Lx} \beta H_y + \beta$ (free energy of small contours)$]$. It is easy, but important, to convince oneself that this is the correct normalization.

From (2.2) one gets that the variance of the field flows as

$$(\varepsilon') = L^{2-d}\varepsilon^2 \tag{2.7}$$

The Imry–Ma argument, based on (2.6) and (2.7), is that the disorder and the temperature are irrelevant for $d > 2$ and so ferromagnetism should persist. However, for $d = 2$ the disorder persists on all scales (while T still goes to zero) and one should not expect the ferromagnetic order to be stable against this random perturbation.

Now we turn to the hierarchical model, where the renormalization group transformation can be carried out exactly. Actually, only the first step (integrate small contours) needs to be done, because on each scale we have essentially only one contour to consider and so there is no blocking.

The hierarchical model is defined by keeping in (2.1) only contours that are boundaries of L^{dn} blocks, and moreover by modifying the compatibility constraint so that all such contours are compatible (i.e., they occur totally independently of each other, at a given scale and between scales). More precisely, on all scales L^n, $n = 0,..., N$ (L^{dN} is the volume of the box) we index the contours γ by C^n_x, the (disjoint) L^{dn}-cubes centered at $L^n x$. The $\gamma(C^n_x)$ has activity $\exp[-(\beta/2d)\,|\partial C^n_x|] = \exp(-\beta\,|\gamma|)$ and $\gamma(C^n_x)$ flips spins in C^n_x. Denoting this constraint on the sum in (2.1) by prime, we may immediately carry out the scale $n = 0$ sum:

$$\begin{aligned} Z(N, h, \beta) &\equiv \sum_{\Gamma}{}' \prod_{\gamma \in \Gamma} e^{-\beta |\gamma|} e^{\beta(h, V^+) - \beta(h, V^-)} \\ &= \sum_{\Gamma^l}{}' \prod_{\gamma \in \Gamma^l} e^{-\beta |\gamma|} e^{\beta(h + \delta h^+, V^+) - \beta(h + \delta h^-, V^-)} \end{aligned} \tag{2.8}$$

where $\delta h_x^{\pm}$ are given by the small contour partition functions:

$$e^{\pm\beta(h+\delta h^{\pm}, Lx)} = \prod_{y \in Lx} (e^{\pm\beta h_y} + e^{-\beta} e^{\mp\beta h_y}) \tag{2.9}$$

(at each $y \in Lx$ we may or may not have a contour that flips the spin at y and has activity $e^{-\beta}$).

In the rhs of (2.8) we have only large contours, i.e., those on scale $n \neq 0$.

Defining

$$\beta_1 = L^{d-1}\beta \tag{2.10}$$

$$h_{1x}^{\pm} = L^{1-d}(h + \delta h^{\pm}, Lx) \tag{2.11}$$

we get

$$\begin{aligned} Z(N, h, \beta) &= Z(N-1, h_1^{\pm}, \beta_1) \\ &\equiv \sum_{\Gamma}{}' \prod e^{-\beta_1 |\gamma|} e^{\beta_1(h_1^-, V^+) - \beta_1(h_1^-, V^-)} \end{aligned} \tag{2.12}$$

and upon iteration

$$Z(N, h, \beta) = Z(N-n, h_n^{\pm}, \beta_n) \tag{2.13}$$

where

$$\beta_n = L^{n(d-1)}\beta \tag{2.14}$$

and $h_n^{\pm}$ satisfy the recursion

$$h_{n+1x}^{\pm} = L^{1-d} \sum_{y \in Lx} \left\{ h_{ny}^{\pm} \pm \frac{1}{\beta_n} \log[1 + e^{-\beta_n} e^{\mp\beta_n(h_{ny}^+ + h_{ny}^-)}] \right\} \tag{2.15}$$

or define $h^+ + h^- = H$ (H here is like the H in ref. 2, but there is a difference of sign in the definition of h^+),

$$Z(N, h, \beta) = e^{\beta_n(h_n^+, V_n)} \tilde{Z}(N-n, H_n, \beta_n) \tag{2.16}$$

$$\tilde{Z}(N-n, H_n, \beta_n) = \sum{}' \prod e^{-\beta_n |\gamma|} e^{-\beta_n(H_n, V^-)} \tag{2.17}$$

and

$$H_{n+1x} = L^{1-d} \sum_{y \in Lx} [H_{ny} + f_n(H_{ny})] \tag{2.18}$$

with

$$f_n(x) = \frac{1}{\beta_n} \log \frac{1 + e^{-\beta_n(1+x)}}{1 + e^{\beta_n(1-x)}} \tag{2.19}$$

It is easy to see that

$$|x + f_n(x)| \leqslant [1 + O(\beta_n^{-1})] \, |x| \tag{2.20}$$

Thus, the free energy of the system is given by the random variable

$$F_N = L^{-Nd} \beta_N h_{N0}^+ \tag{2.21}$$

and, see (2.15),

$$h_{n+1x}^+ = L^{1-d} \sum_{y \in Lx} [h_{ny}^+ + g_n(H_y^n)] \tag{2.22}$$

with

$$g_n(x) = \frac{1}{\beta_n} \log(1 + e^{-\beta(1+x)})$$

Hence we need to control the flow of H_{nx}, which are independent random variables for n fixed. Before embarking on that, let us derive analogous formulas for correlations. Consider first the 1-point function

$$\langle \sigma_0 \rangle_N^+ = Z^{-1} \sum_{\Gamma} \prod e^{-\beta |\gamma|} e^{\beta[(h, V^+) - (h, V^-)]} \sigma_0(\Gamma) \tag{2.23}$$

where

$$\sigma_0(\Gamma) = \begin{cases} \sigma_0^+ = 1 & \text{if} \quad 0 \in V^+ \\ \sigma_0^- = -1 & \text{if} \quad 0 \in V^- \end{cases} \tag{2.24}$$

[zero on the lhs of (2.23) refers to the origin in Z^d, on the rhs to the zeroth step of iteration!]. We get the analogue of (2.23) for the nth step with

$$\sigma_n^{\pm} = (\sigma_{n-1}^{\pm} + \sigma_{n-1}^{\mp} e^{-\beta_n} e^{\mp \beta_n H_{n0}})/(1 + e^{-\beta_n} e^{\mp \beta_n H_{n0}}) \tag{2.25}$$

and then

$$\langle \sigma_0 \rangle_N^+ = \sigma_N^+ \tag{2.26}$$

Let us now analyze (2.18). First let $d > 2$. Assume

$$\langle e^{tH_n} \rangle \leqslant e^{t^2 e_n^2/2} \tag{2.27}$$

Then, since H_{ny} are independent for different y's and f_n is an odd function,

$$\begin{aligned}\langle e^{tH_{n+1}}\rangle &= \langle e^{t'(H_n+f_n)}\rangle^{L^d} \qquad (t' = tL^{1-d})\\ &= \langle \cosh t'(H_n+f_n)\rangle^{L^d}\\ &\leqslant \langle \cosh t'[1+O(\beta_n^{-1})]H_n\rangle^{L^d} \qquad \text{[using (2.20)]}\\ &= \langle \exp\{t'[1+O(\beta_n^{-1})]H_n\}\rangle^{L^d}\\ &\leqslant e^{\varepsilon_{n+1}^2 t^2/2}\end{aligned}$$

with

$$\varepsilon_{n+1}^2 = L^{2-d}[1+O(\beta_n^{-1})]\varepsilon_n^2 \tag{2.28}$$

(2.27) implies that $\mathrm{Prob}(|H_n|>x)\leqslant 2\exp[-(x^2/2\varepsilon_n^2)]$, which, combined with (2.25), easily yields

$$\overline{\sigma_N^+} > 1 - e^{-c\beta} - e^{-c/\varepsilon^2} \tag{2.29}$$

(bar = average over disorder), i.e., the system is indeed ordered for $d>2$.

The $d=2$ case is more interesting. Since $\beta_n\to\infty$ rapidly, let us set $\beta=\infty$ [the errors being negligible, i.e., $O(\beta_n^{-1})$]. Then

$$x+f_\infty(x) = \begin{cases} 1 & x>1\\ x & x\in[-1,1]\\ -1 & x<-1\end{cases} \tag{2.30}$$

and we have to study the recursion relation between random variables given by [see (2.18)]

$$H_{n+1x} = L^{-1}\sum_{y\in Lx}[H_{ny}+f_\infty(H_{ny})] \tag{2.31}$$

We claim that this recursion drives the variance toward zero. Indeed,

$$\begin{aligned}\varepsilon_{n+1}^2 &= \langle H_{n+1}^2\rangle\\ &= \langle [H_n+f_\infty(H_n)]^2\rangle\\ &= \langle H_n^2\rangle - \langle (H_n^2-1)\,\chi((H_n)>1)\rangle\\ &= \langle H_n^2\rangle - \langle (H_n^2-1)\,|\,|H_n|>1\rangle\langle\chi(|H_n|>1)\rangle\\ &= \varepsilon_n^2 - k(\varepsilon_n)\end{aligned} \tag{2.32}$$

$\langle F|A\rangle$ is the expectation of F conditioned on A) where, obviously, if ε_n

did not go to zero, $k(\varepsilon_n)$ would remain bounded away from zero and a contradiction would follow. Thus, $\varepsilon_n \to 0$. To see how fast, we use

$$\langle (H_n^2 - 1) \mid |H_n| > 1 \rangle = O(\varepsilon_n^2)$$

and

$$\langle \chi(|H_n| > 1) \rangle = O(\varepsilon_n) \exp(-\varepsilon_n^{-2}/2) \tag{2.33}$$

which follows from an analysis of (2.31): this recursion keeps H approximately Gaussian but with decreasing variance. By (2.32) and (2.33),

$$\varepsilon_{n+1}^2 = \varepsilon_n^2 [1 - O(\varepsilon_n)\, e^{-(\varepsilon_n^{-2}/2)}]$$

which, to leading order, gives

$$\varepsilon_n^2 \simeq (\log n)^{-1}$$

and

$$(2.33) = O[(n \log n)^{-1}] \tag{2.34}$$

Now we compute the magnetization, averaged over the disorder. First, letting $\beta \to \infty$, one gets from (2.25)

$$\sigma_N^+ = \prod_{n=0}^{N} \alpha(H_{n0}) \tag{2.35}$$

where

$$\alpha(H_{n0}) = \begin{cases} -1 & \text{if} \quad |H_{n0}| > 1 \\ +1 & \text{if} \quad |H_{n0}| < 1 \end{cases}$$

i.e., σ_N changes sign as often as some H_{n0} forces a contour to occur. (We neglect $|H_{n0}| = 1$, which has zero probability.)

If the H_{n0} were independent for different n's, we would have

$$\begin{aligned} \overline{\sigma_N^+} &= \prod_{n=0}^{N} [P(|H_{n0}| < 1) - P(|H_{n0}| > 1)] \\ &= \prod_{n=0}^{N} [1 - 2P(|H_{n0}| > 1)] \end{aligned} \tag{2.36}$$

The H_{n0} are not quite independent of each other, but almost [H_{n0} affects H_{n+1} only by $O(L^{-1})$; see (2.30), (2.31)] and one gets

$$\overline{\sigma_N^+} \simeq \prod_{n=0}^{N} [1 - O(1)\, P(|H_n| > 1)] \tag{2.37}$$

which, by (2.33), (2.34) gives

$$\overline{\sigma_N^+} \simeq \exp\left[-O(1) \sum_{n=1}^{N} (n \log n)^{-1} \right]$$
$$= (\log N)^{-p} \qquad \text{some} \quad p > 0 \tag{2.38}$$

The approach of the magnetization to zero is similar here to the decay of the two-point function, which can be calculated as above, and satisfies

$$\overline{\langle \sigma_0 \sigma_x \rangle} \simeq (\log \log |x|)^{-p} \qquad \text{some} \quad p > 0 \tag{2.39}$$

(since $|x| \sim L^N$).

Let us add some remarks:

1. In the linear approximation to (2.32) ($k = 0$), which is the Imry–Ma argument, contours occur on all scales with a nonzero probability and are almost independent; therefore, infinitely many contours occur and the state is disordered. However, one must observe that, if a contour occurs on a given scale, the field inside it contributes less to the field on the next scale than if there had been no contour. The large fields "screen" themselves by producing contours that surround them. This is the content of (2.30) and (2.31), where a cutoff appears in the contribution of a large field to the next scale. However, as we have seen, all this does is to make contours more rare (but still infinitely many occur) and thus to slow down the decay of correlations.

2. It is interesting to compare the decay of $\overline{\langle \sigma_0 \sigma_x \rangle}$ with that of other correlations. First consider, at fixed $\beta < \infty$, the decay of

$$\langle \sigma_0, \sigma_x \rangle \equiv \langle \sigma_0 \sigma_x \rangle - \langle \sigma_0 \rangle \langle \sigma_x \rangle$$

With probability one (in h) this will be $O(e^{-c\beta |x|})$ (for $d = 2$). Indeed, this measures only thermal fluctuations and the latter are as much suppressed here as in the usual model. One may also consider $\langle \sigma_0 ; \sigma_x \rangle$. We claim that this will be $O(|x|^{-1})$ (up to logarithmic corrections). Indeed, this decay is dominated by that of

$$P(|H_n \pm 1| = O(\beta_n^{-1})) \tag{2.40}$$

with $L^n \cong |x|$.

Indeed, for those events, contours surrounding both 0 and x can occur with large probability in the thermal average. Now (2.40) decays like $\beta_n^{-1} \sim L^{-n} \sim |x|^{-1}$, to leading order.

3. THE RENORMALIZATION GROUP FLOW IN THE ISING MODEL

As we said, there are two steps in the RG transformation of ref. 2: integration of the small contours and blocking of the large ones.

We expect that the first step is roughly similar to the hierarchical model, giving only a correction $O[\varepsilon_n^3 \exp(-1/2\varepsilon_n^2)]$. However, we want to argue that the blocking leads to a bigger correction, and with the opposite sign.

Indeed, in the previous discussion [see Eq. (2.4)] of the blocking, we neglected an important contribution due to the fields in $L\gamma' \equiv \{Lx \mid x \in \gamma'\}$ (i.e., the blocks indexed by γ'). To be precise, $\rho(\Gamma)$ in (2.4) should be multiplied by

$$\exp(\beta((h, V^+(\Gamma) \cap L\gamma') - (h, V^-(\Gamma) \cap L\gamma'))) \tag{3.1}$$

These are the fields inside $L\gamma'$. When one considers $d > 2$, they are irrelevant and can be neglected. To see how they affect the flow in $d = 2$, we have to find how the

$$\min_{\Gamma \subset L\gamma'} (|\Gamma| - (h, V^+(\Gamma) \cap L\gamma') + (h, V^-(\Gamma) \cap L\gamma')) \tag{3.2}$$

behaves. Indeed, this corresponds to the largest term in (2.4) and determines the behavior of $\rho'(\gamma')$. To simplify, let $\Gamma = \{\gamma\}$ consist of one contour and let γ' be sufficiently long. Without the h terms, the minimum is simply the shortest γ in $L\gamma'$ and is of order $L\,|\gamma'|$. However, as observed by Binder,[9] local fluctuations in the h may actually make it favorable for γ to make a detour [that modifies $V^{\pm}(\gamma)$]. Actually, for every part of γ of size $O(\varepsilon^{-2})$, the fluctuations of the fields are of order $\varepsilon O(\varepsilon^{-1}) = O(1)$, and, if they have the right sign, may make kinks [with energy also $O(1)$] in γ favorable. And this reduces the energy by an amount proportional to the number of disjoint subsets of γ of size $O(\varepsilon^{-2})$, i.e., by $O(\varepsilon^2)\,|\gamma|$. So we expect $\rho'(\gamma')$ to have a bound $\exp(-\beta'\,|\gamma'|)$, but with

$$\beta' = \beta L[1 - O(\varepsilon^2)] \tag{3.3}$$

which means that one should normalize H' [see (2.2)] by $L^{-1}[1 - O(\varepsilon^2)]^{-1}$. This, in turn, implies that the variance flows as

$$(\varepsilon')^2 = \varepsilon^2/[1 - O(\varepsilon^2)]^2 \simeq \varepsilon^2 + O(\varepsilon^4) \tag{3.4}$$

(we assume everywhere that ε is small).

This is the basic relation indicating that ε grows under renormalization. If we extrapolate (3.4) to higher ε's, we see that after a number

n of interactions $[n = O(\varepsilon^{-2})]$, ε_n becomes $O(1)$ and there the correlation length is also $O(1)$.[3,4] This leads us to expect a finite correlation length $\xi(\varepsilon) = L^n = \exp O(\varepsilon^{-2})$.

To conclude, the picture of the typical configurations for any $\varepsilon \neq 0$ should be the same as for large ε, but on a scale of order $\xi(\varepsilon)$. That is, there should be intertwined patches of plus and of minus spins, whose diameter is of order $\xi(\varepsilon)$. The boundary between these patches forms a connected (infinite) contour. This is quite different from the picture that emerged from the hierarchical model: a sequence of nested squares with corridors of a given sign. However, this latter picture seems unstable if we allow fluctuations in the shape of the contours and should therefore be regarded as an artifact of the hierarchical model. Let us also mention that another defect of this model is that it does not have a high-temperature, large-field phase in $d > 2$.

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